No-Cloning and No-Deleting theorems through the existence of Incomparable states under LOCC

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No-Cloning and No-Deleting theorems are verified with the constraint on local state transformations via the existence of incomparable states. Assuming the existence of exact cloning or deleting operation defined on a minimum number of two arbitrary states, an incomparable pair of states of the joint system between two parties can be made to compare under deterministic LOCC. We have restricted our proof with the assumption that the machine states of the cloning or deleting operations do not keep any information about the input states. We use the same setting to establish the no-cloning and no-deleting theorems via incomparability that supports the reciprocity of the two operations in their operational senses. The work associates the impossibility of operations with the evolution of an entangled system by LOCC.

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One of the most important task in quantum information processing is to detect the allowable set of operations performed on quantum systems. If someone wants to copy an arbitrary quantum information encoded in a quantum state then no-cloning theorem [1] restricts one to copy arbitrary quantum information exactly. Quite reverse to it, if we want to delete arbitrary quantum information then we have a similar kind of restriction [2, 3]. According to the no-deletion theorem [2, 3], it is not possible to delete arbitrary quantum information encoded in a quantum state to a standard one. On the other hand, manipulation of pure state entanglement provides us some other kind of restrictions on the evolution of quantum systems. Sometimes a specific state may be required to perform a specific information theoretic task. Then Nielsen's criterion [4] determines the possibility of inter-conversion of one pure entangled state shared between two spatially separated parties to another by deterministic LOCC. This result provides us a necessary and sufficient condition for converting a bipartite pure entangled state to another by LOCC with certainty. Now one may ask whether the no-go theorems and other impossibilities only restrict the specific tasks or may be useful for other kind of tasks that seems to be impossible otherwise? To search for a common origin of these impossibilities one have to find the possibility of interconnection between themselves within or outside the quantum formalism. Here we provide a connection between no-cloning and no-deleting theorems with the incomparability of pure entangled states. The work shows, existence of either of the exact cloning or deletion machine that act perfectly on any set of non-orthogonal states, will imply local inter-conversion of a pair of incomparable states with certainty.

We begin with some necessary background to our work. In quantum information theory the no-go theorems are used to define intrinsic properties of quantum systems beyond their usual status of imposing restrictions over the systems. They allow quantum systems to perform some computational tasks which are rather impossible by using classical algorithms. In quantum cryptography [5], the possibility of detecting an eavesdropper having an access on the communication channel emerges out of the well known no-cloning [1] theorem. In terms of information processing, cloning can be viewed as the copying of information encoded in some systems to other systems [6]. If $|\psi\rangle$ be the input state then we describe exact cloning operation as $|\psi\rangle \otimes |b\rangle \Rightarrow |\psi\rangle \otimes |\psi\rangle$, where $|b\rangle$ is some suitably chosen blank state. Now quantum systems will not provide complete accuracy of performing those operations on arbitrary input states. Linearity of quantum operations establishes precisely the impossibility of existence of an Universal Exact Cloning Machine [1, 7]. Unitarity of any quantum evolution also shows that Universal Exact Cloning operation is not physical in nature [8]. Linearity of allowable quantum operations further provides us another constraint which we termed as No-Deletion theorem [2, 3]. Deletion is quite a reverse process than that of cloning. It is performed on two copies of an arbitrary input state and is not possible to delete exactly the information of one copy, keeping intact the information of the other copy. In other words, the operation $|\psi\rangle \otimes |\psi\rangle \Rightarrow |\psi\rangle \otimes |b\rangle$ is not possible exactly for an arbitrary input state $|\psi\rangle$ with certainty.

There are some other no-go theorems defined on single qubit systems, such as the no-flipping theorem [9]. From linearity of quantum operations we find further the restriction of no-partial-cloning, and other no-go theorems obtained from the concepts of various quantum gates [10]. Efforts are made to search the inter-relations between different no-go theorems and relate them with other theories. For example, no-signaling principle restricts any physical operation to evolve in such a way that can not be used to send a signal faster than the speed of light. No-signaling condition preserves all the impossibilities cited above [11, 12, 13, 14]. Again, the constraint of non-increase of entanglement under LOCC, described in a quite similar way as that of the second law of thermodynamics. Applying any local operations on the subsystems of a quantum system together with classical communica-

tions between distant parties, it is impossible to increase the entanglement of the joint system. The no-cloning [15], no-deleting [16], no-flipping [14] and many other impossibilities [16] are connected with this constraint of information theory. Also the interrelation between the cloning and flipping operations is revealed by the conservation laws of simple classical theory [17]. Now a very new kind of information theoretic restriction on allowable quantum operations observed through the existence of incomparable states [18]. This restriction retrieve the no-flipping theorem [19] and also detects impossibility of some general classes of local quantum operations [20]. Here we want to reveal a relation between this constraint with the very famous no-cloning and no-deleting theorems. The work proceeds to verify the reciprocity of the two no-principles by dealing them in a single setting without verifying them separately. The connection between all those no-go theorems of quantum systems with the impossibility of inter-conversion of incomparable states would support the existence of incomparable states beyond their mathematical status from Nielsen's criterion. It provides also the nature of allowable physical operations.

To present our work we need to define first the condition for a pair of states to be incomparable with each other. The notion of incomparability of a pair of bipartite pure entangled states is a consequence of Nielsen's [4] majorization criterion. Suppose we want to convert the pure bipartite state $|\Psi\rangle$ to $|\Phi\rangle$ shared between two parties, say, Alice and Bob by deterministic LOCC. Consider the pair $(|\Psi\rangle, |\Phi\rangle)$ in their Schmidt bases $\{|i_A\rangle, |i_B\rangle\}$ with decreasing order of Schmidt coefficients: $|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\alpha_i}|i_A i_B\rangle$, $|\Phi\rangle = \sum_{i=1}^{d} \sqrt{\beta_i}|i_A i_B\rangle$, where $\alpha_i \geq \alpha_{i+1} \geq 0$ and $\beta_i \geq \beta_{i+1} \geq 0$, for $i = 1, 2, \cdots, d-1$, and $\sum_{i=1}^{d} \alpha_i = 1 = \sum_{i=1}^{d} \beta_i$. The Schmidt vectors corresponding to the states $|\Psi\rangle$ and $|\Phi\rangle$ are $\lambda_\Psi \equiv (\alpha_1, \alpha_2, \cdots, \alpha_d)$ and $\lambda_\Phi \equiv (\beta_1, \beta_2, \cdots, \beta_d)$. Then Nielsen's criterion says $|\Psi\rangle \rightarrow |\Phi\rangle$ is possible with certainty under LOCC if and only if λ_Ψ is majorized by λ_Φ, denoted by $\lambda_\Psi \prec \lambda_\Phi$ and described as,

$$\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i \quad \forall \quad k = 1, 2, \cdots, d \tag{1}$$

It is interesting to note that as a consequence of non-increase of entanglement by LOCC, if $|\Psi\rangle \rightarrow |\Phi\rangle$ is possible under LOCC with certainty, then $E(|\Psi\rangle) \geq E(|\Phi\rangle)$ [where $E(\cdot)$ denote the von-Neumann entropy of the reduced density operator of any subsystem and known as the entropy of entanglement]. Now in case of failure of the above criterion (1), it is usually denoted by $|\Psi\rangle \not\rightarrow |\Phi\rangle$. But it may happen that $|\Phi\rangle \rightarrow |\Psi\rangle$ under LOCC. And if it happens that both $|\Psi\rangle \not\rightarrow |\Phi\rangle$ and $|\Phi\rangle \not\rightarrow |\Psi\rangle$ then we denote it by $|\Psi\rangle \not\leftrightarrow |\Phi\rangle$ and describe $(|\Psi\rangle, |\Phi\rangle)$, as a pair of incomparable states. One of the peculiar feature of the existence of such incomparable pairs is that we are really unable to say which state has a greater amount of entanglement content than that of the other. For 2×2 systems there are no pair of incomparable pure entangled states as described above. For our purpose, we want to mention explicitly the criterion of incomparability for a pair of pure entangled states $|\Psi\rangle, |\Phi\rangle$ of $m \times n$ system where $\min\{m, n\} = 3$. Suppose the Schmidt vectors corresponding to the two states are (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively, where $a_1 > a_2 > a_3 > 0$, $b_1 > b_2 > b_3 > 0$, $a_1 + a_2 + a_3 = 1 = b_1 + b_2 + b_3$. Then it follows from Nielsen's criterion that $|\Psi\rangle, |\Phi\rangle$ are incomparable [18] if and only if, either of the pair of relations

$$\begin{array}{l} a_1 > b_1 \;\;\&\;\; a_3 > b_3 \\ b_1 > a_1 \;\;\&\;\; b_3 > a_3 \end{array} \tag{2}$$

will hold.

Our paper concerns with one-to-two copy exact cloning operation on a minimum number of two arbitrary states $|0\rangle, |\psi\rangle$ in the following form

$$\begin{array}{l} |0\rangle|b\rangle \longrightarrow |0\rangle|0\rangle \\ |\psi\rangle|b\rangle \longrightarrow |\psi\rangle|\psi\rangle \end{array} \tag{3}$$

where $|b\rangle$ is a suitably chosen blank state.

We concentrate entirely within the quantum formalism and for that reason we assume the machine states do not keep any information about the input states. So we drop the machine states in the definition of the cloning operation. No-cloning theorem then turns out to be the impossibility of this operation for arbitrary state $|\psi\rangle$.

Now we consider that Alice and Bob, two spatially separated parties have a particular setting of a pure bipartite state in the form given below

$$\begin{array}{rcl} |\Omega^i\rangle_{AB} & = & \frac{1}{\sqrt{N^i}}\{|1\rangle_A|0\psi 0\psi + \psi 0\psi 0\rangle_B + |2\rangle_A|0\psi\psi 0 \\ & & -\psi 00\psi\rangle_B + |3\rangle_A|00\psi\psi - \psi\psi 00\rangle_B\} \otimes |b\rangle_B \end{array} \tag{4}$$

This is a six particle state where Alice has one qutrit and Bob has four qubits entangled with Alice's system together with a separate qubit in the form of blank state $|b\rangle$. So the joint system is of 3×32 dimension, where $N^i = 2(3 - \alpha^4)$ be the normalizing constant and $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ be an arbitrary qubit with $|\alpha|^2 + |\beta|^2 = 1$. As the arbitrary input state $|\psi\rangle$ can be written in the form $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{-i\phi}\sin\frac{\theta}{2}|1\rangle$, where θ, ϕ satisfy the following equations $0 \leq \phi \leq 2\pi$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, hence without loss of generality, the parameter α is treated here as a real constant.

Tracing out Bob's local system we compute the initial reduced density matrix ρ_A^i on Alice's side in the following form

$$\begin{array}{rcl} \rho_A^i & = & tr_B[\ |\Omega^i\rangle_{AB}\langle\Omega^i|\ ] \\ & = & \frac{1}{N^i}\{2(1 + |\alpha|^4)P[|1\rangle] + 2(1 - |\alpha|^4) \\ & & (P[|2\rangle] + P[|3\rangle])\} \end{array} \tag{5}$$

where $P[|j\rangle] = |j\rangle\langle j|$, for any j. The Schmidt vector of the initial state can be written as $\lambda^i = (\lambda^i{}_1, \lambda^i{}_2, \lambda^i{}_2)$ where $\lambda_1^i = \frac{1+\alpha^4}{3-\alpha^4}$ and $\lambda_2^i = \frac{1-\alpha^4}{3-\alpha^4}$. Hence $\lambda_{max}^i = \max\{\lambda_1^i, \lambda_2^i\} = \lambda_1^i$ and $\lambda_{min}^i = \min\{\lambda_1^i, \lambda_2^i\} = \lambda_2^i$. If the cloning operation defined in equation(3) exists and is

applied on Bob's local system (say on his fourth qubit together with the blank state), the joint pure state shared between Alice and Bob could be exactly transformed to the pure state,

$$\begin{aligned}|\Omega^f\rangle_{AB} \; = \; & \frac{1}{\sqrt{N^f}}\,\{|1\rangle_A|0\psi0\psi\psi+\psi0\psi00\rangle_B+|2\rangle_A|0\psi\psi00 \\ & -\psi00\psi\psi\rangle_B+|3\rangle_A|00\psi\psi\psi-\psi\psi000\rangle_B\}\end{aligned} \tag{6}$$

where $N^f = 2(3-\alpha^5)$ be the normalizing constant. The final reduced density matrix on Alice's side would be

$$\begin{aligned}\rho_A^f \; &= \; tr_B[\,|\Omega^f\rangle_{AB}\langle\Omega^f|\,] \\ &= \; \frac{1}{N^f}\,\{2(1+\alpha^5)P[|1\rangle]+2(1-\alpha^5)(P[|2\rangle]+P[|3\rangle]) \\ &\quad -2\alpha^2(1-\alpha)(|2\rangle\langle 3|+|3\rangle\langle 2|)\}\end{aligned} \tag{7}$$

Hence the Schmidt coefficients of the final state ρ_A^f are $\{\frac{1+\alpha^5}{3-\alpha^5}, \frac{(1+\alpha^2)(1-\alpha^3)}{3-\alpha^5}, \frac{(1-\alpha^2)(1+\alpha^3)}{3-\alpha^5}\}$. If we denote $\lambda_1^f = \frac{1+\alpha^5}{3-\alpha^5}$, $\lambda_2^f = \frac{(1+\alpha^2)(1-\alpha^3)}{3-\alpha^5}$ and $\lambda_3^f = \frac{(1-\alpha^2)(1+\alpha^3)}{3-\alpha^5}$, then $\lambda_{min}^f = \min\{\lambda_1^f, \lambda_2^f, \lambda_3^f\} = \lambda_3^f$ and thus $\lambda_{max}^f = \max\{\lambda_1^f, \lambda_2^f\}$. Now using simple algebra we find, $\lambda_1^f < \lambda_1^i$ and also $\lambda_2^f < \lambda_1^i$ (if, $\lambda_1^f < \lambda_2^f$), which implies that, $\lambda_{max}^f < \lambda_{max}^i$. Finally we get, $\lambda_{min}^f = \lambda_3^f < \lambda_2^i = \lambda_{min}^i$. These inequalities clearly indicate the nature of incomparability of the pair of pure bipartite states $|\Omega^i\rangle$ and $|\Omega^f\rangle$. The incomparability of the states imply that the final state $|\Omega^f\rangle$ can not be achieved from the initial state $|\Omega^i\rangle$ through LOCC with certainty. Thus we are compelled to conclude that the cloning operation performed on Bob's local system to implement the transformation $|\Omega^i\rangle \rightarrow |\Omega^f\rangle$ locally, is not a physical operation. In other words the exact cloning operation is not possible, for any pair of arbitrary non-orthogonal input states. This ensures the successful establishment of no-cloning theorem.

Now if we further treat $|\Omega^f\rangle$ as the initial pure bipartite state, shared between Alice and Bob and assume the existence of an exact deleting machine again defined on only two arbitrary input qubit $|0\rangle, |\psi\rangle$ as

$$\begin{aligned}|0\rangle|0\rangle &\longrightarrow |0\rangle|b\rangle \\ |\psi\rangle|\psi\rangle &\longrightarrow |\psi\rangle|b\rangle\end{aligned} \tag{8}$$

and apply this machine on Bob's local system the joint state $|\Omega^f\rangle$ between them can be converted into the state $|\Omega^i\rangle$. Under the previous arguments it could be easily proved that $|\Omega^i\rangle \not\rightarrow |\Omega^f\rangle$, i.e., the transformation $|\Omega^f\rangle \rightarrow |\Omega^i\rangle$ is impossible by LOCC with certainty. This impossibility directly indicates that the deleting operation defined in equation(8) is not a valid physical operation for arbitrary input states. So this leads us to the formal no-deleting theorem.

In conclusion this work connects the two famous no-go theorems from a new viewpoint that restricts the possible evolution of any quantum system through local operations. It shows the physical reason behind the existence of "*Incomparable Pair of Pure Bipartite States*". This connection makes a bridge between two different aspects of information processing theory. Moreover the most interesting feature is that the no-cloning and no-deleting theorems are treated in the same platform and thus we see the reciprocity of the two theorems from an operational point of view. Although for simplicity we assume that the machine states for both the cloning and deleting operations do not contain any information about the input qubit state, one may not assume this restriction. The result also holds if we consider the general scenario. Another interesting part in our proof is that the state we have considered, has a peculiar kind of symmetry and we require 3×32 dimensional system to prove our result. However one may search for a proof in lower dimensional systems.

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